

# THICKNESS OF THE UNIT SPHERE, $\ell_1$ -TYPES, AND THE ALMOST DAUGAVET PROPERTY

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**ABSTRACT.** We study those Banach spaces  $X$  for which  $S_X$  does not admit a finite  $\varepsilon$ -net consisting of elements of  $S_X$  for any  $\varepsilon < 2$ . We give characterisations of this class of spaces in terms of  $\ell_1$ -type sequences and in terms of the almost Daugavet property. The main result of the paper is: a separable Banach space  $X$  is isomorphic to a space from this class if and only if  $X$  contains an isomorphic copy of  $\ell_1$ .

## 1. INTRODUCTION

For a Banach space  $X$ , R. Whitley [11] introduced the following parameter, called *thickness*, which is essentially the inner measure of non-compactness of the unit sphere  $S_X$ :

$$T(X) = \inf\{\varepsilon > 0: \text{there exists a finite } \varepsilon\text{-net for } S_X \text{ in } S_X\},$$

or equivalently,  $T(X)$  is the infimum of those  $\varepsilon$  such that the unit sphere of  $X$  can be covered by a finite number of balls with radius  $\varepsilon$  and centres in  $S_X$ . He showed in the infinite dimensional case that  $1 \leq T(X) \leq 2$ , and in particular that  $T(C(K)) = 1$  if  $K$  has isolated points and  $T(C(K)) = 2$  if not.

In this paper we concentrate on the spaces with  $T(X) = 2$ . Our main results are the following;  $B_X$  denotes the closed unit ball of  $X$ .

**Theorem 1.1.** *For a separable Banach space  $X$  the following conditions are equivalent:*

- (a)  $T(X) = 2$ ;
- (b) *there is a sequence  $(e_n) \subset B_X$  such that for every  $x \in X$*

$$\lim_{n \rightarrow \infty} \|x + e_n\| = \|x\| + 1;$$

- (c) *there is a norming subspace  $Y \subset X^*$  such that the equation*

$$\|\text{Id} + T\| = 1 + \|T\| \tag{1.1}$$

*holds true for every rank-one operator  $T: X \rightarrow X$  of the form  $T = y^* \otimes x$ , where  $x \in X$  and  $y^* \in Y$ .*

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**Theorem 1.2.** *A separable Banach space  $X$  can be equivalently renormed to have thickness  $T(X) = 2$  if and only if  $X$  contains an isomorphic copy of  $\ell_1$ .*

We mention that it has been proved in [1] that a space with thickness  $T(X) = 2$  contains a copy of  $\ell_1$ .

Recall that a subspace  $Y \subset X^*$  is said to be norming (or 1-norming) if for every  $x \in X$

$$\sup_{y^* \in S_Y} |y^*(x)| = \|x\|.$$

$Y$  is norming if and only if  $S_Y$  is weak\* dense in  $B_{X^*}$ .

Condition (b) of Theorem 1.1 links our investigations to the theory of types [8]. Recall that a type on a separable Banach space  $X$  is a function of the form

$$\tau(x) = \lim_{n \rightarrow \infty} \|x + e_n\|$$

for some bounded sequence  $(e_n)$ . In [8] the notion of an  $\ell_1$ -type is defined by means of convolution of types; a special instance of this is a type generated by a sequence  $(e_n)$  satisfying

$$\tau(x) = \lim_{n \rightarrow \infty} \|x + e_n\| = \|x\| + 1. \quad (1.2)$$

To simplify notation let us call a type like this a *canonical  $\ell_1$ -type* and a sequence  $(e_n) \subset B_X$  satisfying (1.2) a *canonical  $\ell_1$ -type sequence*.

Condition (c) links our investigations to the theory of Banach spaces with the *Daugavet property* introduced in [6] and developed further for instance in the papers [2] [3], [4], [7]; see also the survey [10]. We will say that a Banach space  $X$  has the *Daugavet property with respect to  $Y$*  ( $X \in \text{DPr}(Y)$ ) if the *Daugavet equation* (1.1) holds true for every rank-one operator  $T: X \rightarrow X$  of the form  $T = y^* \otimes x$ , where  $x \in X$  and  $y^* \in Y$ , and it has the *almost Daugavet property* or is an *almost Daugavet space* if it has  $\text{DPr}(Y)$  for some norming subspace  $Y \subset X^*$ . This definition is a generalization (introduced in [5]) of the by now well-known Daugavet property of [6], which is  $\text{DPr}(Y)$  with  $Y = X^*$ .

In this language Theorem 1.2 says, by Theorem 1.1, that a separable Banach space can be renormed to have the almost Daugavet property if and only if it contains a copy of  $\ell_1$ .

In Section 2 we present a characterisation of almost Daugavet spaces in terms of  $\ell_1$ -sequences of the dual. The proofs of Theorems 1.1 and 1.2 will be given in Sections 3 and 4.

The following lemma is the main technical prerequisite that we use; it is the analogue of [6, Lemma 2.2]. Up to part (v) it was proved in [5]; however, (v) follows along the same lines. By a slice of  $B_X$  we mean a set of the form

$$S(y^*, \varepsilon) = \{x \in B_X: \operatorname{Re} y^*(x) \geq 1 - \varepsilon\}$$

for some  $y^* \in S_{X^*}$  and some  $\varepsilon > 0$ , and a weak\* slice  $S(y, \varepsilon)$  of the dual ball  $B_{X^*}$  is a particular case of slice, generated by element  $y \in S_X \subset X^{**}$ .

**Lemma 1.3.** *If  $Y$  is a norming subspace of  $X^*$ , then the following assertions are equivalent.*

- (i)  *$X$  has the Daugavet property with respect to  $Y$ .*

- (ii) For every  $x \in S_X$ , for every  $\varepsilon > 0$ , and for every  $y^* \in S_Y$  there is some  $y \in S(y^*, \varepsilon)$  such that

$$\|x + y\| \geq 2 - \varepsilon. \quad (1.3)$$

- (iii) For every  $x \in S_X$ , for every  $\varepsilon > 0$ , and for every  $y^* \in S_Y$  there is a slice  $S(y_1^*, \varepsilon_1) \subset S(y^*, \varepsilon)$  with  $y_1^* \in S_Y$  such that (1.3) holds for every  $y \in S(y_1^*, \varepsilon_1)$ .
- (iv) For every  $x^* \in S_Y$ , for every  $\varepsilon > 0$ , and for every weak\* slice  $S(x, \varepsilon)$  of the dual ball  $B_{X^*}$  there is some  $y^* \in S(x, \varepsilon)$  such that  $\|x^* + y^*\| \geq 2 - \varepsilon$ .
- (v) For every  $x^* \in S_Y$ , for every  $\varepsilon > 0$ , and for every weak\* slice  $S(x, \varepsilon)$  of the dual ball  $B_{X^*}$  there is another weak\* slice  $S(x_1, \varepsilon_1) \subset S(x, \varepsilon)$  such that  $\|x^* + y^*\| \geq 2 - \varepsilon$  for every  $y^* \in S(x_1, \varepsilon_1)$ .

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## 2. A CHARACTERISATION OF ALMOST DAUGAVET SPACES BY MEANS OF $\ell_1$ -SEQUENCES IN THE DUAL

For the sake of easy notation we introduce two definitions.

**Definition 2.1.** Let  $E$  be subspace of a Banach space  $F$  and  $\varepsilon > 0$ . An element  $e \in B_F$  is said to be  $(\varepsilon, 1)$ -orthogonal to  $E$  if for every  $x \in E$  and  $t \in \mathbb{R}$

$$\|x + te\| \geq (1 - \varepsilon)(\|x\| + |t|). \quad (2.1)$$

**Definition 2.2.** Let  $E$  be a Banach space. A sequence  $\{e_n\}_{n \in \mathbb{N}} \subset B_E \setminus \{0\}$  is said to be an *asymptotic  $\ell_1$ -sequence* if there is a sequence of numbers  $\varepsilon_n > 0$  with  $\prod_{n \in \mathbb{N}} (1 - \varepsilon_n) > 0$  such that  $e_{n+1}$  is  $(\varepsilon_n, 1)$ -orthogonal to  $Y_n := \text{lin}\{e_1, \dots, e_n\}$  for every  $n \in \mathbb{N}$ .

Evidently every asymptotic  $\ell_1$ -sequence is  $1/\prod_{n \in \mathbb{N}} (1 - \varepsilon_n)$ -equivalent to the unit vector basis in  $\ell_1$ , and moreover every element of the unit sphere of  $E_m := \text{lin}\{e_k\}_{k=m+1}^\infty$  is  $(1 - \prod_{n \geq m} (1 - \varepsilon_n), 1)$ -orthogonal to  $Y_m$  for every  $m \in \mathbb{N}$ .

The following lemma is completely analogous to [6, Lemma 2.8]; instead of [6, Lemma 2.1] it uses (v) of Lemma 1.3. So we state it without proof.

**Lemma 2.3.** Let  $Y$  be a norming subspace of  $X^*$ ,  $X \in \text{DPr}(Y)$ , and let  $Y_0 \subset Y$  be a finite-dimensional subspace. Then for every  $\varepsilon_0 > 0$  and every weak\* slice  $S(x_0, \varepsilon_0)$  of  $B_{X^*}$  there is another weak\* slice  $S(x_1, \varepsilon_1) \subset S(x_0, \varepsilon_0)$  of  $B_{X^*}$  such that every element  $e^* \in S(x_1, \varepsilon_1)$  is  $(\varepsilon_0, 1)$ -orthogonal to  $Y_0$ . In particular there is an element  $e_1^* \in S(x_0, \varepsilon_0) \cap S_Y$  which is  $(\varepsilon_0, 1)$ -orthogonal to  $Y_0$ .

We need one more definition.

**Definition 2.4.** A sequence  $\{e_n^*\}_{n \in \mathbb{N}} \subset B_{X^*}$  is said to be *double-norming* if  $\text{lin}\{e_k^*\}_{k=n}^\infty$  is norming for every  $n \in \mathbb{N}$ .

Here is the main result of this section.

**Theorem 2.5.** A separable Banach space  $X$  is an almost Daugavet space if and only if  $X^*$  contains a double-norming asymptotic  $\ell_1$ -sequence.

*Proof.* First we prove the “if” part. Let  $\{e_n^*\}_{n \in \mathbb{N}} \subset B_{X^*}$  be a double-norming asymptotic  $\ell_1$ -sequence, and let  $\varepsilon_n > 0$  with  $\prod_{n \in \mathbb{N}} (1 - \varepsilon_n) > 0$  be such that  $e_{n+1}^*$  is  $(\varepsilon_n, 1)$ -orthogonal to  $Y_n := \text{lin}\{e_1^*, \dots, e_n^*\}$  for every  $n \in \mathbb{N}$ . Let us prove that  $X$  has the Daugavet property with respect to  $Y = \overline{\text{lin}}\{e_n^*\}_{n \in \mathbb{N}}$  where the closure is meant in the norm topology. To do this let us apply (iv) of Lemma 1.3.

Fix an  $x^* \in S_Y$ , an  $\varepsilon > 0$  and a weak\* slice  $S(x, \varepsilon)$  of the dual ball  $B_{X^*}$ . Denote in addition to  $Y_m = \text{lin}\{e_1^*, \dots, e_m^*\}$ ,  $E_m := \text{lin}\{e_k^*\}_{k=m+1}^\infty$ . Using the definition of  $Y$  select an  $m \in \mathbb{N}$  and an  $x_m^* \in Y_m$  such that  $\|x^* - x_m^*\| < \varepsilon/2$  and  $\prod_{n \geq m} (1 - \varepsilon_n) > 1 - \varepsilon/2$ . Since  $E_m$  is norming, there is a  $y^* \in S(x, \varepsilon) \cap S_{E_m}$ . Taking into account that every element of the unit sphere of  $E_m$  is  $(\varepsilon/2, 1)$ -orthogonal to  $Y_m$  we obtain

$$\|x^* + y^*\| \geq \|x_m^* + y^*\| - \|x^* - x_m^*\| \geq 2 - \varepsilon.$$

For the “only if” part we proceed as follows. First we fix a sequence of numbers  $\varepsilon_n > 0$  with  $\prod_{n \in \mathbb{N}} (1 - \varepsilon_n) > 0$  and a dense sequence  $(x_n)$  in  $S_X$ . We can choose these  $x_n$  in such a way that each of them appears in the sequence  $(x_n)$  infinitely many times. Assume now that  $X \in \text{DPr}(Y)$  with respect to a norming subspace  $Y \subset X^*$ . Starting with  $Y_0 = \{0\}$ ,  $\varepsilon_0 = 1$  and applying Lemma 2.3 step-by-step we can construct a sequence  $\{e_n^*\}_{n \in \mathbb{N}} \subset S_Y$  in such a way that each  $e_{n+1}^*$  belongs to  $S(x_n, \varepsilon_n)$  and is  $(\varepsilon_n, 1)$ -orthogonal to  $Y_n$ , where  $Y_n = \text{lin}\{e_1^*, \dots, e_n^*\}$  as before. This inductive construction ensures that the  $e_n^*$ ,  $n \in \mathbb{N}$  form an asymptotic  $\ell_1$ -sequence. On the other hand this sequence meets every slice  $S(x_n, \varepsilon_n)$  infinitely many times, and this implies by density of  $(x_n)$  that  $(e_n^*)$  is double-norming.  $\square$

In Corollary 3.5 we shall observe a somewhat more pleasing version of the last result.

We conclude the section with two examples.

**Proposition 2.6.** *The real space  $\ell_1$  is an almost Daugavet space.*

*Proof.* To prove this statement we will construct a double-norming asymptotic  $\ell_1$ -sequence  $(f_n) \subset \ell_\infty = (\ell_1)^*$ . At first consider a sequence  $(g_n) \subset \ell_\infty$  of elements  $g_n = (g_{n,j})_{j \in \mathbb{N}}$  with all  $g_{n,j} = \pm 1$  satisfying the following independence condition: for arbitrary finite collections  $\alpha_s = \pm 1$ ,  $s = 1, \dots, n$ , the set of those  $j$  that  $g_{s,j} = \alpha_s$  for all  $s = 1, \dots, n$  is infinite (for instance, put  $g_{s,j} := r_s(t_j)$ , where the  $r_s$  are the Rademacher functions and  $(t_j)_{j \in \mathbb{N}}$  is a fixed sequence of irrationals that is dense in  $[0, 1]$ ). These  $g_n$ ,  $n \in \mathbb{N}$ , form an isometric  $\ell_1$ -sequence, and moreover if one changes a finite number of coordinates in each of the  $g_n$  to some other  $\pm 1$ , the independence condition will survive, so the modified sequence will still be an isometric  $\ell_1$ -sequence.

Now let us define the vectors  $f_n = (f_{n,j})_{j \in \mathbb{N}}$ ,  $f_{n,j} = \pm 1$ , in such a way that for  $k = 1, 2, \dots$  and  $n = 2^k + 1, 2^k + 2, \dots, 2^{k+1}$  the vectors  $(f_{n,j})_{j=1}^k \in \ell_\infty^{(k)}$  run over all extreme points of the unit ball of  $\ell_\infty^{(k)}$ , i.e., over all possible  $k$ -tuples of  $\pm 1$ ; for the remaining values of indices we put  $f_{n,j} = g_{n,j}$ . As we have already remarked, the  $f_n$  form an isometric  $\ell_1$ -sequence. Moreover, for every  $k \in \mathbb{N}$  the restrictions of the  $f_n$  to the first  $k$  coordinates form a double-norming sequence over  $\ell_1^{(k)}$ , so  $(f_n)_{n \in \mathbb{N}}$  is a double-norming sequence over  $\ell_1$ .  $\square$

Some ideas of the previous proof will enter into the proof of Theorem 1.1. As a consequence of that theorem, the complex space  $\ell_1$  is almost Daugavet as well. It is worth noting that  $\ell_1$  fails the Daugavet property and cannot even be renormed to have it (see e.g. [6, Cor. 2.7]).

Since  $\ell_\infty$  is isomorphic to  $L_\infty[0, 1]$ , which has the Daugavet property,  $\ell_\infty$  can be equivalently renormed to possess the Daugavet property. Let us show that in the original norm it is not even an almost Daugavet space. This is a special case of the following proposition in which  $\mathbb{K}$  stands for  $\mathbb{R}$  or  $\mathbb{C}$ .

**Proposition 2.7.** *No Banach space of the form  $Z = X \oplus_\infty \mathbb{K}$  is an almost Daugavet space.*

*Proof.* Let us call a functional  $z_0^* \in Z^*$  a *Daugavet functional* if

$$\|\text{Id} + z_0^* \otimes z_0\| = 1 + \|z_0^* \otimes z_0\| \quad \text{for every } z_0 \in Z.$$

We shall show that  $z_0^* = (x_0^*, b_0)$  is not a Daugavet functional if  $b_0 \neq 0$ . Hence all the Daugavet functionals lie in the weak\* closed subspace  $(\{0\} \oplus X)^\perp$  of  $Z^*$ .

So let  $x_0^* \in X^*$  and  $b_0 \neq 0$  with  $\|x_0^*\| + |b_0| = 1$ ,  $z_0^* = (x_0^*, b_0)$  and let  $z_0 = (0, -|b_0|/b_0)$ . If  $z = (x, a) \in B_Z$ , i.e.,  $\|x\| \leq 1$  and  $|a| \leq 1$ , then

$$\begin{aligned} \|z + z_0^*(z)z_0\| &= \max\{\|x\|, |a - z_0^*(z)|b_0|/b_0|\} \\ &\leq \max\{1, |a - (x_0^*(x) + b_0a)|b_0|/b_0|\} \\ &\leq \max\{1, \|x_0^*\| + (1 - |b_0|)\} < 2. \end{aligned}$$

This shows that  $z_0^*$  is not a Daugavet functional.  $\square$

If  $K$  is a compact Hausdorff space with an isolated point, then  $C(K)$  is of the form  $X \oplus_\infty \mathbb{K}$ , hence it fails the almost Daugavet property. But if  $K$  is an uncountable metric space, then  $C(K)$  is isomorphic to  $C[0, 1]$  by Milutin's theorem [12, Th. III.D.19], hence it can be renormed to have the Daugavet property.

### 3. PROOF OF THEOREM 1.1

Since the three properties considered in Theorem 1.1 hold for a complex Banach space  $X$  if and only if they hold for the underlying real space  $X_\mathbb{R}$ , we will tacitly assume in this section that we are dealing with real spaces.

We will accomplish the proof of Theorem 1.1 by means of the following propositions.

The following fact applied for separable spaces is equivalent to implication (c)  $\Rightarrow$  (a) of Theorem 1.1.

**Proposition 3.1.** *Every almost Daugavet space  $X$  has thickness  $T(X) = 2$ .*

*Proof.* Let  $Y \subset X^*$  be a norming subspace with respect to which  $X \in \text{DPr}(Y)$ . According to the definition of  $T(X)$  we have to show that for every  $\varepsilon_0 > 0$  there is no finite  $(2 - \varepsilon_0)$ -net of  $S_X$  consisting of elements of  $S_X$ . In other words we must demonstrate that for every collection  $\{x_1, \dots, x_n\} \subset S_X$  there is a  $y_0 \in S_X$  with  $\|x_k - y_0\| > 2 - \varepsilon_0$  for all  $k = 1, \dots, n$ . But this is an evident corollary of Lemma 1.3(iii): starting with an arbitrary  $y_0^* \in S_{Y^*}$

and applying (iii) we can construct recursively elements  $y_k^* \in S_{Y^*}$  and reals  $\varepsilon_k \in (0, \varepsilon)$ ,  $k = 1, \dots, n$ , in such a way that  $S(y_k^*, \varepsilon_k) \subset S(y_{k-1}^*, \varepsilon_{k-1})$  and

$$\|(-x_k) + y\| > 2 - \varepsilon_0$$

for every  $y \in S(y_k^*, \varepsilon_k)$ . Since  $S(y_n^*, \varepsilon_n)$  is the smallest of the slices constructed, every norm-one element of  $S(y_n^*, \varepsilon_n)$  can be taken as the  $y_0$  we need.  $\square$

For spaces with the Daugavet property the previous proposition has been proved in [9, Prop. 4.1.6].

Let us now turn to the implication (a)  $\Rightarrow$  (b) of Theorem 1.1.

**Proposition 3.2.** *If  $T(X) = 2$  and  $X$  is separable, then  $X$  contains a canonical  $\ell_1$ -type sequence.*

*Proof.* Fix a dense countable set  $A = \{a_n : n \in \mathbb{N}\} \subset S_X$  and a null-sequence  $(\varepsilon_n)$  of positive reals. Since for every  $n \in \mathbb{N}$  the  $n$ -point set  $\{-a_1, \dots, -a_n\}$  is not a  $(2 - \varepsilon_n)$ -net of  $S_X$  there is an  $e_n \in S_X$  with  $\|e_n - (-a_k)\| > 2 - \varepsilon_n$  for all  $k = 1, \dots, n$ . The constructed sequence  $(e_n)$  satisfies for every  $k \in \mathbb{N}$  the condition

$$\lim_{n \rightarrow \infty} \|a_k + e_n\| = \|a_k\| + 1 = 2.$$

By the density of  $A$  in  $S_X$  and a standard convexity argument (cf. e.g. [10, page 78]) this yields that  $(e_n)$  is a canonical  $\ell_1$ -type sequence.  $\square$

By the result in [1] mentioned in the introduction we obtain:

**Corollary 3.3.** *Every almost Daugavet space contains  $\ell_1$ .*

It remains to prove the implication (b)  $\Rightarrow$  (c) of Theorem 1.1.

**Proposition 3.4.** *A separable Banach space  $X$  containing a canonical  $\ell_1$ -type sequence is an almost Daugavet space.*

*Proof.* We will use Theorem 2.5. Fix an increasing sequence of finite-dimensional subspaces  $E_1 \subset E_2 \subset E_3 \subset \dots$  whose union is dense in  $X$ . Also, fix sequences  $\varepsilon_n \searrow 0$  and  $\delta_n > 0$  such that for all  $n$

$$\prod_{k=n}^{\infty} (1 - \delta_k) \geq 1 - \varepsilon_n. \quad (3.1)$$

Passing to a subsequence if necessary we can find a canonical  $\ell_1$ -type sequence  $(e_n)$  satisfying the following additional condition: For every  $x \in \text{lin}(E_n \cup \{e_1, \dots, e_n\})$  and every  $\alpha \in \mathbb{R}$  we have

$$\|x + \alpha e_{n+1}\| \geq (1 - \delta_n)(\|x\| + |\alpha|). \quad (3.2)$$

Then we have for every  $x \in E_n$  and every  $y = \sum_{k=n+1}^M a_k e_k$  by (3.1) and (3.2)

$$\|x + y\| \geq (1 - \varepsilon_n)\|x\| + \sum_{k=n+1}^M (1 - \varepsilon_{k-1})|a_k|. \quad (3.3)$$

Fix a dense sequence  $(x_n)$  in  $S_X$  such that  $x_n \in E_n$  and every element of the range of the sequence is attained infinitely often, that is for each  $m \in \mathbb{N}$  the set  $\{n : x_n = x_m\}$  is infinite. Finally, fix an “independent” sequence  $(g_n) \subset \ell_\infty$ ,  $g_{n,j} = \pm 1$ , as in the proof of Proposition 2.6.

Now we are ready to construct a double-norming asymptotic  $\ell_1$ -sequence  $(f_n^*) \subset X^*$ . First we define  $\tilde{f}_n^*$  on  $F_n := \text{lin}\{x_n, e_{n+1}, e_{n+2}, \dots\}$  by

$$\tilde{f}_n^*(x_n) = 1 - \varepsilon_n, \quad (3.4)$$

$$\tilde{f}_n^*(e_k) = (1 - \varepsilon_{k-1})g_{n,k} \quad (\text{if } k > n). \quad (3.5)$$

By (3.3),  $\|\tilde{f}_n^*\| \leq 1$ , and indeed  $\|\tilde{f}_n^*\| = 1$  by (3.5). Define  $f_n^* \in X^*$  to be a Hahn-Banach extension of  $\tilde{f}_n^*$ . Condition (3.4) and the choice of  $(x_n)$  ensure that  $(f_n^*)$  is double-norming. Let us show that it is an isometric  $\ell_1$ -basis. Indeed, due to our definition of an “independent” sequence, for an arbitrary finite collection  $A = \{a_1, \dots, a_n\}$  of non-zero coefficients the set  $J_A$  of those  $j > n$  that  $g_{s,j} = \text{sign } a_s$ ,  $s = 1, \dots, n$ , is infinite. So by (3.5)

$$\left\| \sum_{s=1}^n a_s f_s^* \right\| \geq \sup_{j \in J_A} \left( \sum_{s=1}^n a_s f_s^* \right) e_j = \sup_{j \in J_A} (1 - \varepsilon_{j-1}) \sum_{s=1}^n |a_s| = \sum_{s=1}^n |a_s|.$$

□

Since we have constructed an isometric  $\ell_1$ -basis (over the reals) in the last proof, we have obtained the following version of Theorem 2.5.

**Corollary 3.5.** *A real separable Banach space  $X$  is an almost Daugavet space if and only if  $X^*$  contains a double-norming isometric  $\ell_1$ -sequence.*

#### 4. PROOF OF THEOREM 1.2

We start with two lemmas.

**Lemma 4.1.** *Let  $X$  be a linear space,  $(e_n) \subset X$ , and let  $p$  be a seminorm on  $X$ . Assume that  $(e_n)$  is an isometric  $\ell_1$ -basis with respect to  $p$ , i.e.,  $p(\sum_{k=1}^n a_k e_k) = \sum_{k=1}^n |a_k|$  for all  $a_1, a_2, \dots \in \mathbb{K}$ . Fix a free ultrafilter  $\mathcal{U}$  on  $\mathbb{N}$  and define*

$$p_r(x) = \mathcal{U}\text{-}\lim_n p(x + r e_n) - r$$

for  $x \in X$  and  $r > 0$ . Then:

- (a)  $0 \leq p_r(x) \leq p(x)$  for all  $x \in X$ ,
- (b)  $p_r(x) = p(x)$  for all  $x \in \text{lin}\{e_1, e_2, \dots\}$ ,
- (c) the map  $x \mapsto p_r(x)$  is convex for each  $r$ ,
- (d) the map  $r \mapsto p_r(x)$  is convex for each  $x$ ,
- (e)  $p_r(tx) = t p_{r/t}(x)$  for each  $t > 0$ ,
- (f)  $|p_r(x) - p_r(y)| \leq p(x - y)$  for all  $x, y \in X$ .

*Proof.* The only thing that is not obvious is that  $p_r$  is positive; note that (b) is a well-known property of the unit vector basis of  $\ell_1$ . Now, given  $\varepsilon > 0$  pick  $n_\varepsilon$  such that

$$p(x + r e_{n_\varepsilon}) \leq \mathcal{U}\text{-}\lim_n p(x + r e_n) + \varepsilon.$$

Then for each  $n \neq n_\varepsilon$

$$\begin{aligned} p(x + r e_n) &= p(x + r e_{n_\varepsilon} + r(e_n - e_{n_\varepsilon})) \\ &\geq 2r - p(x + r e_{n_\varepsilon}) \\ &\geq 2r - \mathcal{U}\text{-}\lim_n p(x + r e_n) - \varepsilon; \end{aligned}$$

hence  $2\mathcal{U}\text{-}\lim_n p(x + r e_n) \geq 2r - 2\varepsilon$  and  $p_r \geq 0$ . □

**Lemma 4.2.** *Assume the conditions of Lemma 4.1. Then the function  $r \mapsto p_r(x)$  is decreasing for each  $x$ . The quantity*

$$\bar{p}(x) := \lim_{r \rightarrow \infty} p_r(x) = \inf_r p_r(x)$$

*satisfies (a)–(c) of Lemma 4.1 and moreover*

$$\bar{p}(tx) = t\bar{p}(x) \quad \text{for } t > 0, x \in X. \quad (4.1)$$

*Proof.* By Lemma 4.1(a) and (d),  $r \mapsto p_r(x)$  is bounded and convex, hence decreasing. Therefore,  $\bar{p}$  is well defined. Clearly, (4.1) follows from (e) above.  $\square$

Since for separable spaces the condition  $T(X) = 2$  is equivalent to the presence of a canonical  $\ell_1$ -type sequence and a canonical  $\ell_1$ -type sequence evidently contains a subsequence equivalent to the canonical basis of  $\ell_1$ , to prove Theorem 1.2 it is sufficient to demonstrate the following:

**Theorem 4.3.** *Let  $X$  be a Banach space containing a copy of  $\ell_1$ . Then  $X$  can be renormed to admit a canonical  $\ell_1$ -type sequence. Moreover if  $(e_n) \subset X$  is an arbitrary sequence equivalent to the canonical basis of  $\ell_1$  in the original norm, then one can construct an equivalent norm on  $X$  in such a way that  $(e_n)$  is isometrically equivalent to the canonical basis of  $\ell_1$  and  $(e_n)$  forms a canonical  $\ell_1$ -type sequence in the new norm.*

*Proof.* Let  $Y$  be a subspace of  $X$  isomorphic to  $\ell_1$ , and let  $(e_n)$  be its canonical basis. To begin with, we can renorm  $X$  in such a way that  $Y$  is isometric to  $\ell_1$  and  $(e_n)$  is an isometric  $\ell_1$ -basis.

Let  $\mathcal{P}$  be the family of all seminorms  $\tilde{p}$  on  $X$  that are dominated by the norm of  $X$  and for which  $\tilde{p}(y) = \|y\|$  for  $y \in Y$ . By Zorn's lemma,  $\mathcal{P}$  contains a minimal element, say  $p$ . We shall argue that

$$\lim_{n \rightarrow \infty} p(x + e_n) = p(x) + 1 \quad \forall x \in X. \quad (4.2)$$

To show this it is sufficient to prove that for every free ultrafilter  $\mathcal{U}$  on  $\mathbb{N}$

$$\mathcal{U}\text{-}\lim_n p(x + e_n) = p(x) + 1 \quad \forall x \in X. \quad (4.3)$$

To this end associate to  $p$  and  $\mathcal{U}$  the functional  $\bar{p}$  from Lemma 4.2. Note that  $0 \leq \bar{p} \leq p$ , but a priori  $\bar{p}$  need not be a seminorm. However,

$$q(x) = \frac{\bar{p}(x) + \bar{p}(-x)}{2}$$

in the real case, resp.

$$q(x) = \int_0^1 \bar{p}(e^{2\pi i t} x) dt$$

in the complex case, defines a seminorm, and  $q \leq p$ . (Lemma 4.1(f) implies that the integrand is a continuous function of  $t$ .) By Lemma 4.1(b) and by minimality of  $p$  we get that

$$q(x) = p(x) \quad \forall x \in X. \quad (4.4)$$

Now, since  $p(x) = p(\lambda x) \geq \bar{p}(\lambda x)$  whenever  $\lambda$  is a scalar of modulus 1, (4.4) implies that  $p(x) = \bar{p}(x)$ . Finally, by Lemma 4.1(a) and the definition of  $\bar{p}$  we have  $p(x) = p_r(x)$  for all  $r > 0$ ; in particular  $p(x) = p_1(x)$ , which is our claim (4.3).



To complete the proof of the theorem, consider

$$\|x\| := p(x) + \|[x]\|_{X/Y};$$

this is the equivalent norm that we need. Indeed, clearly  $\|x\| \leq 2\|x\|$ . On the other hand,  $\|x\| \geq \frac{1}{3}\|x\|$ . To see this assume  $\|x\| = 1$ . If  $\|[x]\|_{X/Y} \geq \frac{1}{3}$ , there is nothing to prove. If not, pick  $y \in Y$  such that  $\|x - y\| < \frac{1}{3}$ . Then  $p(y) = \|y\| > \frac{2}{3}$ , and

$$\|x\| \geq p(x) \geq p(y) - p(x - y) > \frac{2}{3} - \|x - y\| > \frac{1}{3}.$$

Therefore,  $\|\cdot\|$  and  $\|\cdot\|$  are equivalent norms, and

$$\lim_{n \rightarrow \infty} \|x + e_n\| = \lim_{n \rightarrow \infty} p(x + e_n) + \|[x]\|_{X/Y} = p(x) + 1 + \|[x]\|_{X/Y} = \|x\| + 1$$

shows that  $(e_n)$  is a canonical  $\ell_1$ -type sequence for the new norm.  $\square$

We would like to mention another proof of Theorem 4.3 that was suggested to us by W.B. Johnson. In this proof  $X$  is a real Banach space. Let again  $Y \subset X$  be a subspace isometric to  $\ell_1$  with canonical basis  $(e_n)$ . We denote by  $(r_n)$  the sequence of Rademacher functions in  $L_\infty[0, 1]$ . Then there is a norm-1 operator from  $Y$  to  $L_\infty[0, 1]$  mapping  $e_n$  to  $r_n$ , for each  $n$ . Since  $L_\infty[0, 1]$  is 1-injective, the operator can be extended to a norm-1 operator  $T: X \rightarrow L_\infty[0, 1]$ . If we let

$$\|x\| = \|Tx\| + \|[x]\|_{X/Y},$$

then this equivalent norm works; the details of the computation are the same as above.

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